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On the relative importance of Taylor–vortex and non-axisymmetric modes in flow between rotating cylinders

By E. R. KRUEGER,† A. GROSS‡ AND R. C. DI PRIMA
Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York

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The small-gap equations for the stability of Couette flow with respect to non-axisymmetric disturbances are derived. The eigenvalue problem is solved by a direct numerical procedure. It is found that there is a critical value of $\Omega_2/\Omega_1(\Omega_1, \Omega_2$ and R_1, R_2 are the angular velocities and radii of the inner and outer cylinders respectively) of approximately $-0{\cdot}78$, above which the critical disturbance is axisymmetric and below which it is non-axisymmetric. In particular for $R_1/R_2 = 0{\cdot}95$, $\Omega_2/\Omega_1 = -1$, the wave-number in the azimuthal direction of the critical disturbance is $m = 4$. This result is confirmed when the full linear disturbance equations are considered, i.e. the small-gap approximation is not made.

1. Introduction

The stability of a viscous flow between two concentric rotating cylinders (Couette flow) was first considered by Taylor (1923). In this classical paper, he observed experimentally that the flow becomes unstable at sufficiently high speeds of the inner cylinder, the instability yielding a steady secondary motion in the form of cellular toroidal vortices (Taylor vortices) spaced regularly along the axis of the cylinder; his theoretical predictions were in excellent agreement with his observations.

The linearized problem for the stability of Couette flow with respect to axisymmetric disturbances leads to an eigenvalue problem for the determination of the critical speed of the inner cylinder; the latter appears in the form of a Taylor number T (based on the speed of the inner cylinder and containing a geometric factor representating the curvature effect), which is a function of the parameters $\mu = \Omega_2/\Omega_1, \eta = R_1/R_2$, and the dimensionless axial wave-number a of the disturbance. Here Ω_1, Ω_2 and R_1, R_2 are the angular velocities and radii of the inner and outer cylinders respectively. For the most part, recent theoretical work has dealt with the development of more elegant and practical techniques for solving the eigenvalue problem and with computations for a wider range of parameters than originally considered by Taylor, for example, the effect of gap

† Present address: Department of Applied Mathematics, University of Colorado, Boulder, Colorado.
‡ Present address: Bell Telephone Laboratories, Murray Hill, New Jersey.

size and large negative values of μ. More general problems which arise from the addition of axial or circumferential pressure gradients and/or the addition of axial, circumferential, or radial magnetic fields and/or the use of non-Newtonian fluids have also been considered. Much, but certainly not all, of this work is discussed in a comprehensive treatise by Chandrasekhar (1961); see also a brief survey paper by Di Prima (1963). In all these investigations, the stability of the basic flow has been considered only with respect to axisymmetric disturbances.

In this paper we will consider the stability of Couette flow with respect to non-axisymmetric disturbances. There are several reasons for considering this problem. First, from a purely mathematical point of view such an investigation is necessary in order to complete the analysis of the stability of Couette flow. (See Lin 1955.)

Secondly, it is known from experiment that non-axisymmetric disturbances play an important role in the instability of Taylor vortices. Typically for $\mu = 0$ it has been observed by Taylor (1923), Schultz-Grunow & Hein (1956), Coles (1960, 1965), Schwarz, Springett & Donnelly (1964), and Nissan, Nardacci & Ho (1963) that with increasing speed of the inner cylinder above the critical speed the vigour of the circulation in the Taylor vortices at first increases, but eventually a second critical speed is reached at which the vortices assume a 'wavy' form in the circumferential direction and move with a certain wave velocity in that direction. In the mathematical analysis of this phenomenon based on the full non-linear equations, as suggested by Di Prima & Stuart (1964), it is necessary to know the critical speeds, amplification rates, and the corresponding eigenfunctions for non-axisymmetric disturbances as predicted by linear theory. While we will not discuss this phenonenon further in this paper, one might call attention to the particularly detailed and complete experimental account given by Coles (1965) of the flow development and possible states of motion in one particular set of concentric cylinders.

Thirdly, and most important for the present analysis, are the indications in the literature that for μ sufficiently negative the critical speed for Couette flow may occur for non-axisymmetric disturbances rather than for axisymmetric disturbances. The experimental observations in regard to this point are somewhat contradictory. Taylor's (1923) observations for the three sets of cylinders $\eta = 0{\cdot}74$, $0{\cdot}88$ and $0{\cdot}94$ and for a wide range of positive and negative values of μ indicate that the original instability of Couette flow leads to a symmetric flow. However, he does point out that while for μ numerically less than a certain positive number, which appears to vary with η, the vortex motion is stable with increasing speed of the inner cylinder; for $\eta \geqslant 0{\cdot}88$ and $\mu < -1$ on the other hand, 'the symmetric rings of coloured fluid which invariably appeared in the first instance if the experiment was carefully performed, were found to break up shortly afterwards'. Lewis (1928), using sets of cylinders with $\eta = 0{\cdot}53$, $0{\cdot}70$ and $0{\cdot}76$, found in all three cases that for values of μ less than approximately $-0{\cdot}4$ pulsating motions (non-axisymmetric motions) occurred at speeds either above critical or at critical. 'There appears to be no consistency in the speeds at which the pulsing type of motion sets in; sometimes the motion changed straightway at the critical speed to the pulsating type.' Unfortunately it is impossible to determine from the data given by Lewis for what particular values of η and μ the latter situation occurred. The observations of Donnelly & Fultz (1960) for the case $\eta = \frac{1}{2}$ are in general agreement with those of Taylor. They note that at values of $-\mu$ between $\frac{1}{5}$ and 1 the cells break up spontaneously within less than a minute of formation. They do not discuss the subsequent motion.

Nissan *et al.* (1963), using a set of cylinders with $\eta = 0{\cdot}85$, have measured the critical speed for Taylor vortices and the second critical speed at which the Taylor vortex motion breaks down into a non-axisymmetric motion. With decreasing μ the value of the second critical speed approaches that for Taylor vortices, the two points coinciding at $\mu = -0{\cdot}73$. Further, the authors observe, 'In all experiments where μ was between $-0{\cdot}70$ and $-0{\cdot}75$ only wavy vortices could be produced; steady non-wavy vortices were entirely absent.' Coles (1965), using an apparatus with $\eta = 0{\cdot}88$ (but with a short axial length compared to that of most experimenters), has depicted the Taylor instability boundary (singly periodic flow) and a second boundary for doubly periodic flows for a wide range of values of μ (see figures 2 (a–c) of his paper). In contrast to the observations of Nissan *et al.* Coles observes that the doubly periodic régime lies above the Taylor régime for $\mu \simeq -1$, though extrapolating from figure 2(b) it is possible that the boundaries may cross at $\mu \simeq -3$. However, he noted (p. 399) that for opposite rotation of the two cylinders a weak spiral configuration (see figure 15(c)) is quite typical of the Taylor instability boundary except at low Reynolds numbers for the outer cylinder. Snyder (Brown University) in a private communication has informed the authors that in some preliminary experiments with $\eta = \frac{1}{2}$ he finds that for $\mu \simeq -1$ the lowest mode of instability is a non-axisymmetric one, apparently a weak helical motion similar to that observed by Coles. While there are a number of other experimental papers in which results are quoted for negative values of μ, in so far as the authors know there is no additional pertinent discussion of the form of the motion that the instability takes.

In part these differences in the experimental observations may be due to the different geometries, and to the different methods of visualization or measurement. While the authors are certainly not qualified to discuss such matters, we might note, for the record, that Taylor and Donnelly & Fultz used a visual technique observing the motion of dye traces. Lewis, Coles, and Nissan, Nardacci & Ho used a visual technique, observing the motion of fine aluminium particles suspended in the fluid. The latter also observed the flow by dispersing small droplets of ethyl alcohol and water in mineral oil. Snyder used a thermistor anemometer (see Lambert, Karlsson & Snyder 1964) to detect the vortex motion.

The mathematical problem of the stability of Couette flow for non-axisymmetric disturbances was first considered by Di Prima (1961). Using the Galerkin method he solved the resulting eigenvalue problem for the case of small-gap ($\eta \to 1$) and $\mu \geqslant 0$. The results show, as expected, that the critical speed increases with increasing wave-number in the azimuthal direction, the minimum corresponding to axisymmetric disturbances. A somewhat surprising fact, in view of the known stability of the vortex motion for $\mu \geqslant 0$ and T slightly greater than the critical value, is that the critical Taylor number for non-axisymmetric disturbances is only slightly greater than that for axisymmetric disturbances. That non-axisymmetric motions do not occur immediately with increasing T can only be explained by considering the full non-linear equations, as has been done by Di Prima & Stuart (1964). More recently, the results of Di Prima (1961) have been confirmed by the direct numerical calculations of Roberts (1965) who considered the case $\mu = 0$, but did not make the small-gap assumption. He gives results for $\eta = 0{\cdot}95$, $0{\cdot}90$, $0{\cdot}85$ and $0{\cdot}75$. In all cases the critical Taylor number increases with the azimuthal wave-number. Krueger (1962) extended Di Prima's analysis to the case $\mu < 0$. His preliminary computations using Galerkin's method indicated that in the small-gap case, for $\mu \simeq -0{\cdot}8$ or less, non-axisymmetric disturbances would occur at lower Taylor numbers than those required for the growth of axisymmetric disturbances. Since these computations were not complete enough to determine the critical values of the axial and azimuthal wave-numbers, the more complete investigation reported here was necessary. The preliminary results obtained by Krueger are confirmed by the present analysis; for η near one and $\mu \simeq -0{\cdot}78$ the critical Taylor number occurs for non-axisymmetric disturbances!

In § 2 the stability problem is derived in the case for which the gap between the cylinders is small compared to the radius of the inner cylinder. A numerical procedure for solving the eigenvalue problem is described in § 3, and the results of the numerical computations are discussed in § 4. To check the validity of the conclusions concerning the occurrence of non-axisymmetric disturbances which are based on the small-gap equations, the full set of linear disturbance equations is considered in § 5.

2. The eigenvalue problem

Let r, θ, and z denote the usual cylindrical polar co-ordinates, and let u_r, u_θ, and u_z denote the components of velocity in the increasing r-, θ- and z-directions respectively. Consider two infinitely long concentric circular cylinders with the z-axis as their common axis. Let the radii and angular velocities of the inner and outer cylinders be R_1, R_2 and Ω_1, Ω_2 respectively. The equations of motion for a viscous incompressible fluid admit the exact steady solution, Couette flow,

$$u_r = u_z = 0, \quad u_\theta = V(r) = Ar + (B/r), \tag{1}$$

where

$$A = (\Omega_2 R_2^2 - \Omega_1 R_1^2)/(R_2^2 - R_1^2), \quad B = (\Omega_1 - \Omega_2) R_1^2 R_2^2/(R_2^2 - R_1^2). \tag{2}$$

To study the stability of this flow we superimpose a general disturbance on the basic solution, substitute in the equations of motion and the continuity equation and neglect quadratic terms. Since the coefficients in these equations depend only upon r, it is possible to look for solutions of the form†

$$u_\theta = V(r) + v(r)\, e^{i(\omega t + m\theta + \lambda z)}, \tag{3}$$

with similar expressions for the other components of velocity and the pressure. In order to insure that the solution is bounded as $z \to \pm\infty$ and single-valued, it is

† It is understood that when the calculations are completed we take the real part of equation (3).

necessary that λ be real and m be an integer. Without loss of generality, we can take m to be zero or a positive integer. The parameter ω will, in general, be complex.

In this section we will be concerned with the small-gap problem in which the gap $d = R_2 - R_1$ is small compared to R_1 so that terms $O(d/R_1)$ can be neglected. The derivation of the small-gap equations is essentially the same as for the classical Taylor problem, except that now we must consider terms involving differentiation with respect to the circumferential co-ordinate θ. In the limit $d/R_1 \to 0$ the distance around the cylinders becomes infinitely large compared to the gap between the cylinders, and the angular co-ordinate must be replaced by a suitable unbounded continuous variable. The proper procedure for doing this is as follows. Consider the second momentum equation

$$\frac{\partial u_\theta}{\partial t} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial \theta} + \ldots = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \ldots\right), \tag{4a}$$

or by substituting from equation (3) and neglecting quadratic terms

$$i[\omega + m\Omega(r)]\, v + \ldots = \nu\left(\frac{d^2 v}{dr^2} + \ldots\right), \tag{4b}$$

where $\Omega(r) = V(r)/r$. It is natural to use Ω_1 as a reference angular velocity, and d as a reference length. Then scaling t in units d^2/ν, we see that the second term in equations (4) is of apparent scale $\Omega_1 d^2/\nu$ as compared with other terms in the equations. There are two possible limits. The first, with curvature effects completely neglected, would correspond to the problem of the stability of shear flow between parallel plates, and the Reynolds number $R = \Omega_1 R_1 d/\nu$ would be held fixed as $d/R_1 \to 0$. In this case, if the second term in equations (4) were to be retained, then $\partial/\partial\theta$ would need to introduce a factor $(d/R_1)^{-1}$ or equivalently $m(d/R_1)$ would need to remain finite† as $d/R_1 \to 0$. In the second limit, however, with which we are concerned, we wish to retain curvature effects. Recall that for the classical Taylor problem, while curvature effects are neglected in many places when $d/R_1 \ll 1$, they are retained through the centrifugal force terms by requiring that the Taylor number T, which is proportional to $(\Omega_1 R_1 d/\nu)^2 (d/R_1)$, be held fixed as $d/R_1 \to 0$. In this case, if the second term in equations (4) is to be retained, then $\partial/\partial\theta$ must introduce a factor $(d/R_1)^{-\frac{1}{2}}$ or, equivalently, $m(d/R_1)^{\frac{1}{2}}$ must remain finite as $(d/R_1) \to 0$. This scaling has been used by Krueger & Di Prima (1962), Bisshopp (1963*a*), and by Krueger (1962). It has also been suggested by Bisshopp (1963*b*) in his investigations of different small-gap limits.

Thus we introduce the dimensionless variables

$$\left.\begin{aligned} r = R_1 + xd, \quad \delta = d/R_1, \quad \mu = \Omega_2/\Omega_1, \\ a = \lambda d, \quad \sigma = \omega d^2/\nu, \quad k = (-\Omega_1/4A)^{\frac{1}{2}} m, \quad T = -4A\Omega_1 d^4/\nu^2. \end{aligned}\right\} \tag{5}$$

Notice, since $A = -\Omega_1(1-\mu)/2\delta$ plus terms $O(1)$, we have asymptotically

$$k \sim [\delta/2(1-\mu)]^{\frac{1}{2}} m, \quad T \sim 2(1-\mu)(\Omega_1 R_1 d/\nu)^2 \delta, \tag{6}$$

which more clearly shows the scaling of the azimuthal wave-number and the

† Note $m\theta = (m/R_1)(R_1\theta) = \alpha s$, where $\alpha = m/R_1$, $s = R_1\theta$ and the wave-number α is scaled with respect to the gap width d.

form of the Taylor number T. Finally, eliminating the perturbations in the pressure and the axial velocity, letting

$$v = R_1\Omega_1 v', \quad u = \frac{\nu}{d}\frac{\Omega_1}{2A\delta}u', \tag{7}$$

where u is proportional to the radial component of velocity;† and then letting $\delta \to 0$ with a, k, σ, and T held fixed, we obtain the following sixth-order system of homogeneous equations

$$\left.\begin{aligned} L(D^2-a^2)\,u &= -a^2T\Omega_l(x)\,v, \\ Lv &= u. \end{aligned}\right\} \tag{8}$$

We have dropped the primes on u and v, and

$$L \equiv D^2 - a^2 - i[\sigma + k\sqrt{(T)}\,\Omega_l(x)], \quad D \equiv d/dx, \quad \Omega_l(x) = 1-(1-\mu)\,x. \tag{9}$$

The boundary conditions at $x = 0$ and $x = 1$ are

$$u = Du = v = 0. \tag{10}$$

Equations (8) are identical with those derived by Krueger (1962). They differ by one term from those considered by Di Prima (1961), which are not formally correct in the present small-gap limit. However, for the range of parameters considered by Di Prima, the term retained is extremely small and does not introduce any error in the results quoted there. Notice that while δ does not appear explicitly in equations (8), (9) or (10), it plays an important role through its appearance in the definition of k. In our mathematical limit $\delta \to 0$, the wave-number k must be treated as a continuous parameter. However, for a given value of $\delta \ll 1$ it has physical meaning only for values corresponding to positive integer or zero values of the azimuthal wave-number m. Finally, note that we would obtain precisely equations (8) if we had assumed that the disturbance velocities were of the form $v(r)\exp[i(\omega t + m\theta)]\cos\lambda z$, i.e. a wave standing in the axial direction but travelling in the azimuthal direction. On the other hand it is not possible to find solutions of the linearized disturbance equations corresponding to standing waves in the azimuthal direction. Within the framework of the present theory it is of course impossible to decide between travelling and standing waves in the axial direction; we will return to this point in §4.

The homogeneous set of equations (8) with the boundary conditions (10) determine an eigenvalue problem of the form

$$F(\mu, a, k, \sigma, T) = 0. \tag{11}$$

The marginal state is characterized by the imaginary part of σ, σ_i, equal to zero. For a given value of μ, which determines the basic velocity up to a scale factor, we wish to determine the minimum real positive value of T over all real $a > 0$ and real $k \geqslant 0$, for which there is a solution of equation (11) with $\sigma_i = 0$. This value of T, $T_c(\mu)$, is the critical value of T for the assigned value of μ. For values of $T > T_c(\mu)$ there will exist solutions of equations (8) and (10) for certain

† This is the same scaling that is used in the classical small-gap problem for axisymmetric disturbances.

values of a and k with $\sigma_i < 0$ and the disturbance will grow exponentially. The values of a and k corresponding to $T_c(\mu)$ determine the form of the critical disturbance. Note that if the critical value of k is not zero, we must for a given geometry (assigned value of $\delta \ll 1$) look at the discrete set of values of k corresponding to $m = 0, 1, 2, \ldots$ to determine $T_c(\mu)$. For this reason, for an assigned value of μ, it is necessary to compute the critical value of T for a wide enough range of values of k to cover the different possibilities. We will denote by $T_c(\mu, k)$ the critical value of T for assigned values of μ and k. The corresponding value of the real part of σ, σ_r determines the frequency of the oscillation. For a fixed z the wave will propagate in the direction of the basic flow with an angular velocity (in units of Ω_1) given by $c = -\omega_r/m\Omega_1 = -\sigma_r/k\sqrt{T}$. Here r used as a subscript denotes the real part.

3. Method of solution

The two-point eigenvalue problem defined by equations (8) and (10) is difficult to treat analytically. The system of equations (8) is really a twelfth-order system of real equations with variable coefficients. While in theory the Galerkin method can be used, it is necessary to take several terms (more with decreasing μ) in the series for u and v, and the complex algebra becomes rather tedious. It is more convenient to use direct numerical procedures such as those used by Di Prima (1955), Harris & Reid (1964), Sparrow, Munro & Jonsson (1964), and Roberts (1965) for similar hydrodynamic stability problems. The procedure has been described by Harris & Reid and will only be briefly summarized here.

Let us first rewrite the system of equations (8) as a system of first-order equations:

$$\left.\begin{aligned} DU = V, \quad DV = Y + aU, \quad DW = X, \\ DX = M(x)\,W + U, \quad DY = Z, \quad DZ = -a^2T\Omega_l(x)\,W + M(x)\,Y, \end{aligned}\right\} \tag{12}$$

where
$$M = a^2 + i[\sigma + k\sqrt{(T)}\,\Omega_l(x)], \quad U = u, \quad V = Du,$$
$$W = v, X = Dv, \quad Y = (D^2 - a^2)\,u.$$

The boundary conditions at $x = 0$ and $x = 1$ are

$$U = V = W = 0. \tag{13}$$

A set of three linearly-independent solutions of the system of differential equations (12) which satisfy the boundary condition at $x = 0$ can be constructed by imposing the initial conditions

$$(U_j, V_j, W_j, X_j, Y_j, Z_j) = \left\{\begin{matrix} (0,0,0,1,0,0) & \text{for} & j = 1, \\ (0,0,0,0,1,0) & \text{for} & j = 2, \\ (0,0,0,0,0,1) & \text{for} & j = 3. \end{matrix}\right\} \tag{14}$$

Any solution of the system of equations (12) satisfying the boundary conditions (13) at $x = 0$ can be represented as a linear combination of these three solutions. A necessary condition that this linear combination also satisfies the boundary conditions $U = V = W = 0$ at $x = 1$ is the vanishing of the determinant

$$F(\mu, a, k, \sigma, T) \equiv \begin{vmatrix} U_j(1) \\ V_j(1) \\ W_j(1) \end{vmatrix} = 0. \tag{15}$$

This is the required characteristic equation. The marginal state is determined by setting $\sigma_i = 0$. For assigned values of μ, a and k we wish to determine the minimum real positive value of T and the corresponding real value of σ for which equation (15) is satisfied. The minimum of the set of values of T over all real positive a determines the critical value of T, $T_c(\mu, k)$, for the assigned values of μ and k.

We proceed as follows. First remember that the solutions

$$\mathbf{U}_j = (U_j, V_j, W_j, X_j, Y_j, Z_j)$$

are complex-valued; hence, equation (15), with $\sigma_i = 0$, gives two real-valued equations of the form

$$G(\mu, a, k, \sigma_r, T) = 0, \quad H(\mu, a, k, \sigma_r, T) = 0. \tag{16}$$

To determine a root of these equations for fixed μ, a and k, we choose three pairs of trial points in the $T-\sigma_r$ plane. For each of these three points, the fundamental set of solutions $\mathbf{U}_j$ are obtained by integrating the system of first-order equations (12) by the Runge–Kutta method. Then the functions G and H are evaluated, and bivariate interpolation is used to obtain a new approximation to the root of equations (16). Iteration is continued until the root is approximated with sufficient accuracy. For all the computations reported in this paper the iteration process converged when the starting values were in a sufficiently small neighbourhood of the root. Once the root was determined, the process was repeated for a sufficient number of values of a so that the minimum of T with respect to a, $T_c(\mu, k)$ could be determined. Quadratic polynomial interpolation with an interval in a of 0·02 was used to determine $T_c(\mu, k)$ and the corresponding values of a and σ_r. Finally, the entire process was repeated for several values of k so that a curve of $T_c(\mu, k)$ versus k could be drawn for the assigned value of μ.

It is worth saying a few words about the error in the various numerical procedures, particularly since for μ near -1 some rather interesting and new results are found corresponding to variations in $T_c(\mu, k)$ with k of 5 % or less. First, all computations were carried out in fixed point arithmetic on an IBM 1410 computer with all variables allotted eight integer places and twelve decimal places. The determinant F was evaluated by pivotal condensation and the root of equations (16) was determined to at least six significant figures. With these precautions, it is estimated that the round-off error is negligible in comparison with the truncation error. The truncation error can be estimated by noting that the Runge–Kutta method is of order h^4, where h is the step size, and then using the method of Richardson's deferred approach to the limit. We assume, of course, that the error in evaluating $F(\mu, a, \sigma_r, k, T)$ is negligible. Several checks were run using step sizes of $h = 0·05$, $0·025$ and $0·0125$. For example, for the extreme case $\mu = -1·25$, $k = 0·79057$ and $a = 4·20$ the corresponding values of T were 29,727·60; 29,735·14 and 29,735·51 respectively. The extrapolated 'exact' values of T using the first two and the last two results are 29,735·64 and 29,735·54 respectively. The results of this and other checks indicate that the maximum error in any of the tabulated values of T_c, a and σ_r in table 1 is not more than ± 2 in the fourth significant figure, and that a step size of $h = 0·05$ is satisfactory. The error in most cases is probably considerably less. For the classical Taylor problem, ($k = 0$, $\sigma = 0$), the present results agree with those given by Harris & Reid (1964) to within ± 1 in the last digit for a, and through four significant figures for T. Finally, as is well known (see Harris & Reid 1964, or Sparrow, *et al.* 1964) there is the danger that, with the large values of T that are found with decreasing

μ	k	a	$T_c(\mu, k)$	$-\sigma_r$	c	$T_c(\mu, k)/$ $T_c(\mu, 0)$
0	0	3·127	3390·1	0	0	1·0000
	0·15811	3·131	3402·5	4·8534	0·5262	1·0037
	0·31623	3·143	3440·3	9·7661	0·5265	1·0148
	0·47434	3·163	3504·8	14·799	0·5270	1·0338
	0·63246	3·190	3598·6	20·020	0·5277	1·0615
	0·79057	3·225	3725·6	25·505	0·5285	1·0990
−0·75	0	3·406	10519	0	0	1·0000
	0·15811	3·417	10560	5·8138	0·3578	1·0039
	0·31623	3·451	10726	11·612	0·3546	1·0196
	0·47434	3·514	11114	17·575	0·3515	1·0565
	0·63246	3·605	11846	24·090	0·3500	1·1261
	0·79057	3·730	13099	31·780	0·3512	1·2453
	0·90000	3·842	14434	38·331	0·3545	1·3721
−0·80	0	3·493	11795	0	0	1·0000
	0·15811	3·489	11783	6·3041	0·3673	0·9989
	0·31623	3·499	11840	12·376	0·3597	1·0038
	0·47434	3·547	12152	18·418	0·3522	1·0302
	0·63246	3·636	12888	24·954	0·3475	1·0926
	0·79057	3·769	14267	32·763	0·3470	1·2095
	1·00000	4·041	17806	47·456	0·3556	1·5096
−1·00	0	3·999	18669	0	0	1·0000
	0·11180	3·941	18478	6·3375	0·4170	0·9898
	0·15811	3·888	18296	8·8326	0·4130	0·9800
	0·22361	3·799	17970	12·127	0·4046	0·9626
	0·31623	3·690	17489	16·235	0·3882	0·9368
	0·33541	3·675	17404	17·013	0·3845	0·9323
	0·44721	3·642	17129	21·297	0·3639	0·9175
	0·47434	3·647	17132	22·316	0·3594	0·9176
	0·55901	3·686	17345	25·600	0·3477	0·9291
	0·63246	3·745	17810	28·716	0·3402	0·9540
	0·79057	3·940	19900	37·115	0·3328	1·0659

TABLE 1. The critical value of T and the corresponding values of a, σ_r, and c for assigned values of μ and k

μ, the solutions $\mathbf{U}_j$ of equation (12) will become linearly dependent near $x = 1$. For the range of μ considered here, $\mu > -1{\cdot}25$, we did not encounter this difficulty. A more detailed discussion of the numerical procedures and the error estimates can be found in Gross (1964).

4. Results

In table 1 the critical values of $T_c(\mu, k)$ and a, as well as the corresponding values of the dimensionless frequency σ_r, and the dimensionless angular wave velocity $c = -\sigma_r/k\sqrt{T}$ are tabulated for $\mu = 0$, $-0{\cdot}75$, $-0{\cdot}80$, and $-1{\cdot}0$ and a suitable range of assigned values of k. (A more complete table (table A) covering the range $-1{\cdot}25 \leqslant \mu < 1$ from which figures 1–3 were constructed has been lodged with the editor.) The choice of the values of k is such that for any reasonable value of δ, say $\delta < \frac{1}{10}$, the critical value of T for different values of m can be found by interpolation from the values tabulated in table 1. Also in the last column, the ratio $T_c(\mu, k)/T_c(\mu, 0)$ is given. For μ greater than approximately $-0{\cdot}78$, $T_c(\mu)$ occurs

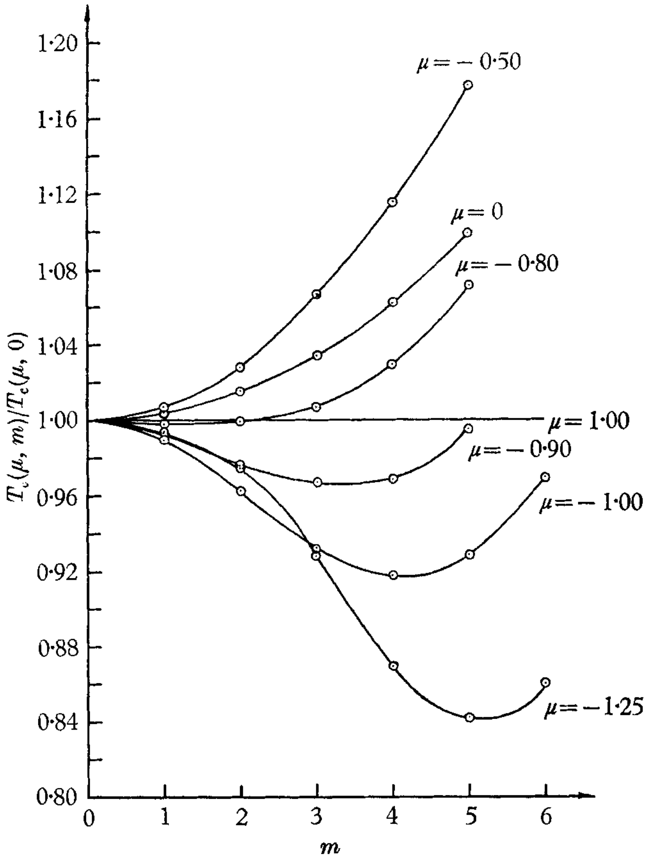

FIGURE 1. The variation of $T_c(\mu, m)/T_c(\mu, 0)$ with $m(\delta = \frac{1}{20})$ for assigned values of μ.

for $k = 0$, i.e. the critical disturbance is axisymmetric. On the other hand, for μ less than approximately $-0{\cdot}78$, $T_c(\mu)$ occurs for $k \neq 0$, indicating that the critical disturbance may be non-axisymmetric.† In practice, of course, the determination of $T_c(\mu)$ in these cases depends upon the discrete set of values of $T_c(\mu, k)$ corresponding to the values $m = 0, 1, 2, 3, \ldots$ for the assigned δ. However, if $T_c(\mu)$ occurs for $k = k_c \neq 0$, it is always possible by using (see equation (6))

$$\delta = 2k^2(1-\mu)/m^2, \tag{17}$$

† It is satisfying to note that Prof. Coles, in a private communication, has indicated that for a set of cylinders with $\delta = 0{\cdot}144$ the change from closed vortex rings to a weak helical structure on the Taylor boundary (see the introduction to the present paper) occurs for $-\mu$ in the range $0{\cdot}75$–$0{\cdot}80$.

to choose $\delta \ll 1$ and m so that the critical disturbance will be non-axisymmetric. We will return to this point later in this section.

To interpret the results in as simple a manner as possible, it is convenient to choose a definite value of δ say $\delta = \frac{1}{20}$, which is a reasonable value for small-gap experimental work. We denote the appropriate Taylor numbers (since δ is fixed) by $T_c(\mu, m)$. In figure 1, the values of $T_c(\mu, m)/T_c(\mu, 0)$ for different values of m with $\delta = \frac{1}{20}$ are given for several values of μ. For convenience the points corresponding to each value of μ are connected by a continuous curve. Note

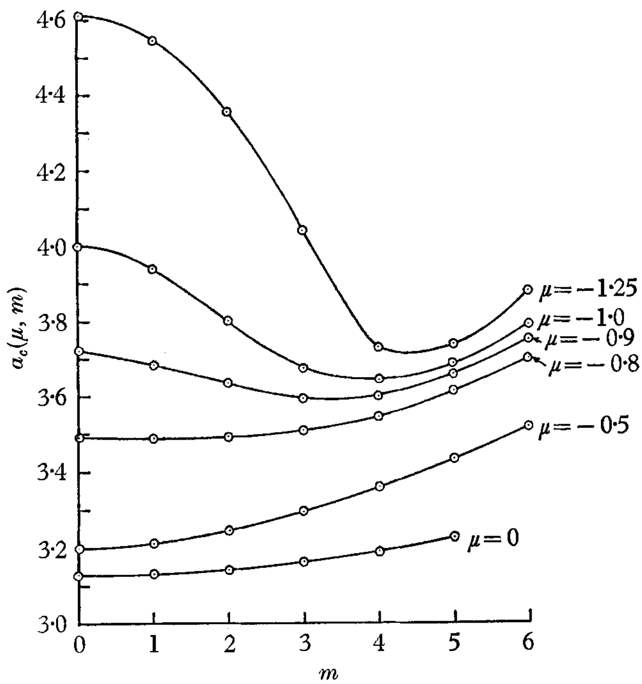

FIGURE 2. The variation of $a_c(\mu, m)$ with $m(\delta = \frac{1}{20})$ for assigned values of μ.

that for $\mu = -0{\cdot}80$, $-0{\cdot}90$, $-1{\cdot}00$ and $-1{\cdot}25$, the critical value of T corresponds to a non-axisymmetric disturbance with 1, 3, 4 and 5 waves in the azimuthal direction, respectively. The corresponding critical values of a, $a_c(\mu, m)$ for different values of $m(\delta = \frac{1}{20})$ for assigned values of μ are shown graphically in figure 2. In the cases for which non-axisymmetric disturbances occur, the critical value of a is less than the critical value of a for an axisymmetric disturbance. Thus the axial wavelength, $2\pi/\lambda = 2\pi d/a$, for non-axisymmetric disturbances will be slightly greater than the value predicted for an axisymmetric disturbance at that value of μ.

In figure 3, the variation of $T_c(\mu, m)/T_c(\mu, 0)$ with μ for assigned values of $m(\delta = \frac{1}{20})$ is shown. For a given value of μ, the critical Taylor number is given by the lowest point on the set of curves. Thus in the ranges (determined only approximately from figure 3) $-0{\cdot}78 < \mu$, $-0{\cdot}81 < \mu < -0{\cdot}78$, $-0{\cdot}84 < \mu < -0{\cdot}81$, $-0{\cdot}93 < \mu < -0{\cdot}84$, $-1{\cdot}13 < \mu < -0{\cdot}93$, $? < \mu < -1{\cdot}13$, the critical Taylor number corresponds to disturbances with $m = 0$, 1, 2, 3, 4 and 5 respectively.

Hence for fixed $\delta \ll 1$, the azimuthal wave-number m of the critical disturbance appears to be a monotone increasing function (though not continuously, of course) of $-\mu$. Further, with decreasing μ, $\mu < -0{\cdot}78$, the points at which the critical value of m jumps discontinuously to the next higher value occur quite quickly at first, but later more slowly. For example, at $\mu = -0{\cdot}78$, $m_c = 1$; at $\mu = -0{\cdot}81$, $m_c = 2$; at $\mu = -0{\cdot}84$, $m_c = 3$; but we must go to $\mu = -0{\cdot}93$ for $m_c = 4$.†

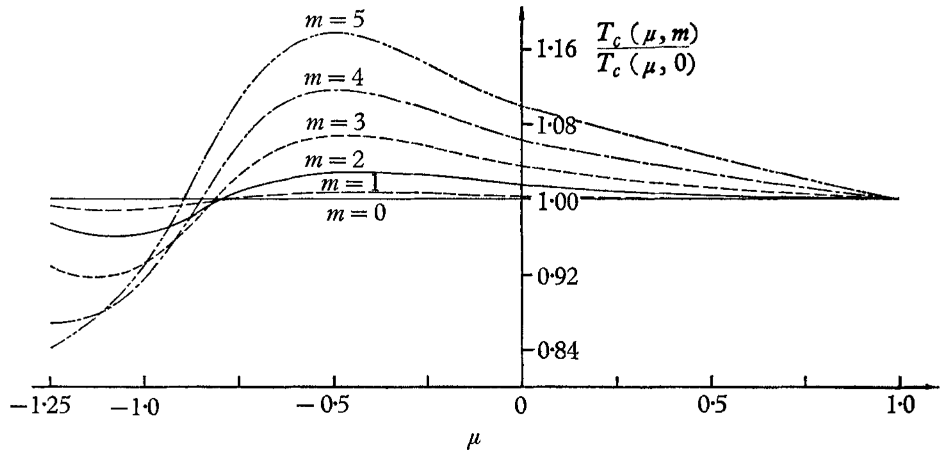

FIGURE 3. The variation of $T_c(\mu, m)/T_c(\mu, 0)$ with μ for assigned values of $m(\delta = \frac{1}{20})$.

Now let us return to equation (17). For a fixed value of $\mu < -0{\cdot}78$, equation (17) shows that it is possible to increase the azimuthal wave number of the critical disturbance by decreasing δ and, of course, vice versa. Typically for $\mu = -1$, $k_c = 0{\cdot}44721$ and from equation (17) $\delta = 0{\cdot}800/m_c^2$. Thus, corresponding to $\delta = 0{\cdot}800$, $0{\cdot}200$, $0{\cdot}089$, $0{\cdot}050$ and $0{\cdot}032$ we find $m_c = 1$, 2, 3, 4 and 5 respectively.‡ Clearly the first two results are not meaningful since the small-gap approximation could hardly be expected to be valid for such values of δ. However, the decrease of m_c with increasing δ is in qualitative agreement with the results obtained in the next section using the full linear disturbance equations.

Of interest are the differences in the critical speed of the inner cylinder for axisymmetric and non-axisymmetric disturbances ($m = 0, 1, 2, \ldots$) for different values of μ. For $\delta = \frac{1}{20}$, and $\mu = -1$, the critical value of $R = \Omega_1 R_1 d/\nu$ occurs for $m = 4$ and we have $[R_c(\mu, 0) - R_c(\mu, 4)]/R_c(\mu, 0) \simeq 0{\cdot}04$, which is a small percentage change.§ For $\mu = -0{\cdot}8$ and $-0{\cdot}9$ the changes are even smaller. Such small changes are probably within or nearly within the limits of experimental errors for many of the experiments that have been performed. Thus, assuming

† For other values of δ similar computations can be made. For an assigned value of μ the values of k corresponding to different values of m are computed and the corresponding values of T are determined by interpolation with the use of the entries in table 1. The minimum value of T determines the critical value of m for the given values of δ and μ.

‡ For intermediate values of δ, the critical value of m can be determined by the procedure outlined in the previous footnote.

§ A table (table B) of critical values of R for different values of μ and m for $\delta = \frac{1}{20}$ has been lodged with the editor of the *Journal*, for consultation by interested readers.

that the critical non-axisymmetric disturbances are stable for increasing $T > T_c$ and hence can be expected to exist physically, it may be that they could not be detected by critical speed measurements alone.

While the changes in the computed parameter T are larger, for example $T_c(\mu)/T_c(\mu, 0) = 0{\cdot}9175$ for $\mu = -1$, nevertheless they are still small enough that they could arise spuriously from the small-gap approximation. For this reason the full linear disturbance equations are considered in the next section. The results confirm the present finding. However, they do show that the small-gap approximation, even for $\delta = \frac{1}{20}$, introduces rather large, but apparently uniform in k or m, errors in the critical value of T for negative values of μ.

Once a root of equation (15) is determined, the corresponding eigenfunction can be computed up to a multiplicative constant. For the particular cases $\mu = 0$, $k = 0$, $0{\cdot}15811$, and $0{\cdot}31623$; and $\mu = -1$, $k = 0$, $0{\cdot}22361$, and $0{\cdot}47434$ the eigenfunctions have been tabulated by Gross (1964).

For $\mu = 0$, the critical Taylor number occurs for $k = 0$, $\sigma_r = 0$, the corresponding eigenfunction $u(x)$, $v(x)$ is real-valued, the disturbance is independent of time, and we can expect a steady axisymmetric secondary motion (Taylor vortices) for T slightly greater than T_c. Assuming that the amplitude of the fundamental $u(x)\cos\lambda z$, $v(x)\cos\lambda z$ is A, A small, then Davey (1962) has shown that the effect of the non-linear terms is to introduce a first harmonic $O(A^2)$ and a correction to the radial dependence of the fundamental $O(A^3)$ and so on. Thus for T slightly greater than T_c, the solution of the linearized problem at $T = T_c$ gives a good description, within a multiplicative factor, of the steady motion that is observed.

On the other hand, for $\mu = -1$, the critical condition corresponds to $k \neq 0$, $\sigma_r \neq 0$ and we can expect an unsteady non-axisymmetric secondary motion. In addition, the eigenfunction $u(x)$ and $v(x)$ is complex-valued and will only be determined up to a complex-valued constant which will introduce a phase angle in the disturbance. More generally when $k \neq 0$ it is impossible, within the framework of linear stability theory, to distinguish between a solution of the disturbance equations corresponding to a wave travelling in the θ-direction but standing in the z-direction ($\exp[i(\omega t + m\theta)]\cos\lambda z$), and a wave travelling in both the θ- and z-directions ($\exp[i(\omega t + m\theta + \lambda z)]$). Note that in either case there will be a phase angle dependent on x since the eigenfunction is complex-valued. In addition, the unknown complex-valued multiplicative factor will introduce an amplitude and a phase angle.

The experimental evidence of Coles (1965) and Snyder (private communication) indicates that for $\mu \simeq -1$ and T slightly greater than T_c, there is a weak helical vortex motion corresponding to a wave travelling in both the θ- and z-directions. It is probable that with increasing T above T_c several equilibrium states are reached, the first being proportional to $\exp[i(\omega t + m\theta + \lambda z)]$ and the last consisting of a number of disturbances which allow the existence of wavy vortices such as those observed by Taylor (1923), by Coles (1965) in the régime he describes as doubly periodic, and by others. For example, a suitable combination of disturbances which are proportional to $\exp[i(\omega t + m\theta)]\cos\lambda z$ and $\exp[i(\omega t + m\theta)]\sin\lambda z$ can lead to wavy vortices. In this regard, the generation of the harmonics of the disturbance may play an important role; also disturbances with different wave-numbers may appear. To resolve such points it is necessary to consider the full non-linear equations. (For the case $\mu = 0$, see Di Prima & Stuart 1964.) It would be helpful to have more detailed experimental evidence concerning the critical state and the subsequent growth of the disturbance for $\mu = -1$. We should also bear in mind that in any experimental apparatus the theoretical conditions of infinitely long cylinders cannot be realized.

5. The wide-gap problem

In this section we consider the linearized problem for the stability of Couette flow with respect to non-axisymmetric disturbances without making the small-gap approximation. The notation here is slightly different than in the previous sections. The velocity and pressure perturbations, u_r', u_θ', u_z' and p' are given by

$$\begin{aligned} \{u_r', u_\theta', u_z'\} &= R_1\Omega_1\{u(x), v(x), w(x)\}\exp[i(\omega t + m\theta + \lambda z)], \\ p' &= \rho\nu\Omega_1\pi(x)\exp[i(\omega t + m\theta + \lambda z)]. \end{aligned} \tag{18}$$

It is convenient to eliminate the pressure perturbation $\pi(x)$ by introducing the variable $X(x)$ defined by

$$\pi(x) = D^*u(x) - X(x), \tag{19}$$

where
$$D^* = d/dx + \xi(x), \quad \xi(x) = \delta/(1+\delta x), \quad \delta = d/R_1. \tag{20}$$

Then letting $Y = D^*v$ and $Z = Dw$, and making use of the continuity equation to eliminate D^*u, we obtain the following system of six first-order equations

$$\left.\begin{aligned} D^*u &= -im\xi(x)\,v - iaw, \\ D^*v &= Y, \quad Dw = Z, \\ DX &= M(x)\,u + 2[im\xi^2(x) - \surd(T)\,\Omega^*(x)]\,v, \\ DY &= [M(x) + m^2\xi^2(x)]\,v - im\xi(x)\,X + ma\xi(x)\,w - 2[im\xi^2(x) - A^*\surd T]\,u, \\ D^*Z &= [M(x) + a^2]\,w - iaX + am\xi(x)\,v, \end{aligned}\right\} \tag{21}$$

where
$$M(x) = a^2 + m^2\xi^2(x) + i[\sigma + m\surd(T)\,\Omega^*(x)],$$

$$\Omega^*(x) = A^* + B^*\xi^2(x), \quad A^* = -\frac{1}{2}\left(\frac{-A}{\Omega_1}\right)^{\frac{1}{2}}, \quad B^* = \frac{B}{2d^2}(-A\Omega_1)^{-\frac{1}{2}},$$

and a, σ, and T are defined in equations (5).

The boundary conditions are $u = v = w = 0$ at $x = 0$ and 1.

The set of equations (21) are the same as those considered by Roberts (1965), except for changes in notation. However, his calculations are limited to $\mu = 0$.

The eigenvalue problem for marginal stability, $F(\mu, \delta, a, m, \sigma_r, T) = 0$, can be solved in precisely the same manner as that described in §3. Computations have been carried out† for $\eta = R_1/R_2 = 1/(1+\delta) = 0{\cdot}95$ ($\delta = 0{\cdot}052632$), for a range of values of μ from 0 to -2, and for $\mu = -1$ for a range of values of η from 0·95 to 0·60. The critical value of T and the corresponding values of m, a, σ_r, and

† Through an oversight $\eta = 0{\cdot}95$ instead of $\delta = \frac{1}{20}$ was used. However, this causes no difficulty since the corresponding results for small-gap theory can be found easily by interpolation in table 1 for $\delta = 0{\cdot}052632$. Most of the computations for table 2 were done on an IBM 7094II at the Bell Telephone Laboratories.

$c = -\omega_r/m\Omega_1$ are tabulated in table 2. (A more complete table (table C) used in constructing figures 4 and 5 may be seen on application to the editor of the *Journal*).

For comparison, the results for $T_c(\mu, m)$ using the small-gap equations and using the exact equations for $\eta = 0{\cdot}95$ are displayed in figures 4 and 5 for $\mu = 0$ and $\mu = -1$, respectively. Notice that for $\mu = 0$, for which the critical distur-

η	μ	m	a	T_c	$-\sigma_r$	c	$T_c/T_c(m=0)$
0·95	0	0	3·128	3509·9	0		1
	−0·80	3	3·561	13730	15·106	0·3591	0·9865
	−1·0	4	3·680	20072	23·358	0·3641	0·8966
	−1·25	5	3·774	30632	33·102	0·3555	0·8363
	−1·50	6	4·002	45307	43·616	0·3391	0·8056
	−1·75	6	3·986	65411	51·537	0·3504	0·7864
	−2·0	7	4·483	91298	64·147	0·3310	0·7681
0·95	−1	4	3·680	20072	23·358	0·3641	0·8966
0·90	—	3	3·721	23861	26·896	0·3583	0·8789
0·85	—	3	3·847	29130	34·330	0·3341	0·8736
0·80	—	2	3·835	36767	33·009	0·3674	0·8833
0·75	—	2	3·873	46243	40·086	0·3523	0·8717
0·70	—	2	3·984	60099	48·472	0·3380	0·8665
0·65	—	2	4·177	81079	58·974	0·3251	0·8681
0·60	—	2	4·456	114043	72·626	0·3135	0·8775

TABLE 2. The critical value of T and the corresponding values of m, a, σ_r and c for assigned values of μ and η

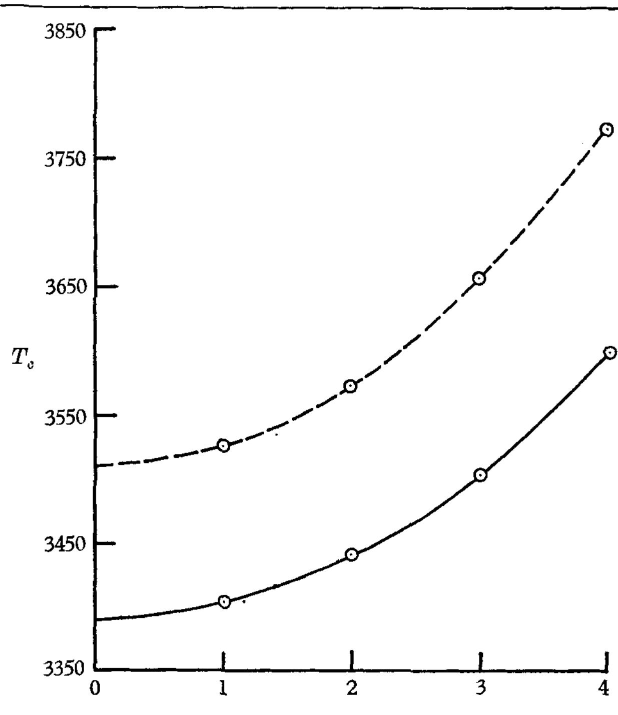

FIGURE 4. The variation of T_c with m for $\mu = 0$, $\eta = 0{\cdot}95$. ———, Small-gap approximation; – – – – –, wide-gap.

bance is axisymmetric, and for $\mu = -1$, for which the critical disturbance is non-axisymmetric, the qualitative agreement between the results using the small-gap approximation and the exact equations is excellent. The 'curves' for $T_c(\mu, m)$ are almost exactly parallel to each other. On the other hand, the correction for gap size can be considerable for μ negative even for a value of δ as small as 0·052632. For $\mu = 0$, $\eta = 0{\cdot}95$ and $m = 0$, the ratio of the exact value of T_c to that computed using the small-gap approximation is 1·04; for $\mu = -1$ the corresponding ratio is 1·20. The corresponding curves for $a_c(\mu, m)$

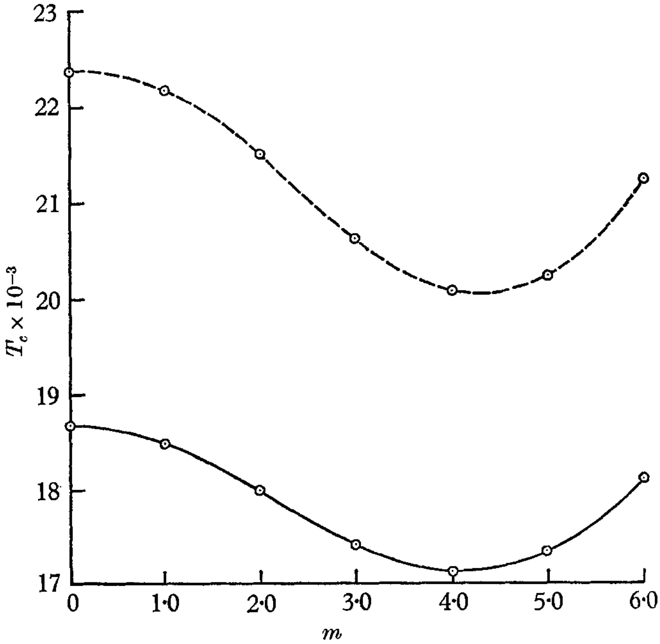

FIGURE 5. The variation of T_c with m for $\mu = -1$, $\eta = 0{\cdot}95$. ———, Small-gap approximation; – – – – –, wide-gap.

are also nearly parallel, and here the correction for gap size at $\mu = -1$ is not so significant. (For $\eta = 0{\cdot}95$ and $\mu = 0$ and -1, a table (table D) comparing the critical values of T, a and c using small-gap theory and using the full equations for assigned values of m has been lodged with the editor.)

The computations using the exact linear stability equations (see table 2) show the following: (1) for δ small (certainly if $\eta > 0{\cdot}6$) and μ sufficiently negative, the critical Taylor number will correspond to a non-axisymmetric disturbance. In particular, for $\eta = 0{\cdot}95$ and $\mu = -1$ the critical disturbance is predicted to have an azimuthal wave-number $m = 4$, which agrees with the prediction of small-gap theory. (2) For $\eta = 0{\cdot}95$, and undoubtedly for other values of η, the critical value of m is non-decreasing with decreasing μ, again in agreement with the prediction of small-gap theory. (3) For $\mu = -1$ the critical value of m is non-increasing with decreasing η† (i.e. increasing δ), at least for values of η

† This is probably true for any value of μ, but is only of interest when $\mu < -0{\cdot}78$, in which case the critical disturbance is non-axisymmetric in the limit as $\eta \to 1$, (i.e. $\delta \to 0$).

down to $\eta = 0{\cdot}6$. The critical value of m jumps from 4 to 3 at about $\eta \simeq 0{\cdot}918$, and from 3 to 2 at $\eta \simeq 0{\cdot}820$. It may be possible, by decreasing η below $0{\cdot}6$, to reach a point at which the critical Taylor number would occur for an axisymmetric disturbance, i.e. to stabilize the flow to non-axisymmetric disturbances. However, we did not investigate this point further. Because of the excessive computations involved, no attempt was made to determine a dividing curve in the $(\mu\eta)$-plane corresponding to states for which the critical disturbance is axisymmetric or non-axisymmetric.

6. Summary

It is usually assumed in the literature that the critical Taylor number for the stability of Couette flow will correspond to an axisymmetric disturbance. The results of this paper show that this assumption is not correct, for sufficiently negative values of $\mu = \Omega_2/\Omega_1$. The analysis based on the small-gap disturbance equations shows that for μ less than a value of about $-0{\cdot}78$ the critical Taylor number corresponds to a non-axisymmetric disturbance provided that δ is sufficiently small. For $\mu < -0{\cdot}78$ the critical value of the azimuthal wave-number increases with decreasing μ for fixed δ, and increases with decreasing δ for fixed μ. The variation is of course not continuous since the azimuthal wave-number takes on only integer values. The variation in the critical Taylor number for non-axisymmetric disturbances, compared to that for axisymmetric disturbances, is small. However, the variation as predicted by the small-gap theory is confirmed by a consideration of the full linear stability equations. On the other hand, the correction for gap size at $\mu = -1$ is quite large, even for such small values of δ as $0{\cdot}05$. When the critical disturbance is a non-axisymmetric one, the corresponding wave-number in the axial direction is slightly less than for an axisymmetric disturbance (Taylor vortex) for the same values of η and μ (see figure 2).

As this work was being submitted for publication, the authors learned in a private communication from Snyder, also see Synder & Karlsson (1965), that a number of the theoretical results given here had been verified experimentally. In particular for $\eta = 0{\cdot}958$ the secondary motion is axisymmetric for μ greater than about $-0{\cdot}766$, and non-axisymmetric (apparently of a weak helical structure) for μ less than this value. In addition, the wave-numbers in the azimuthal direction for different values of μ are in remarkably good agreement with the present theoretical predictions.

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